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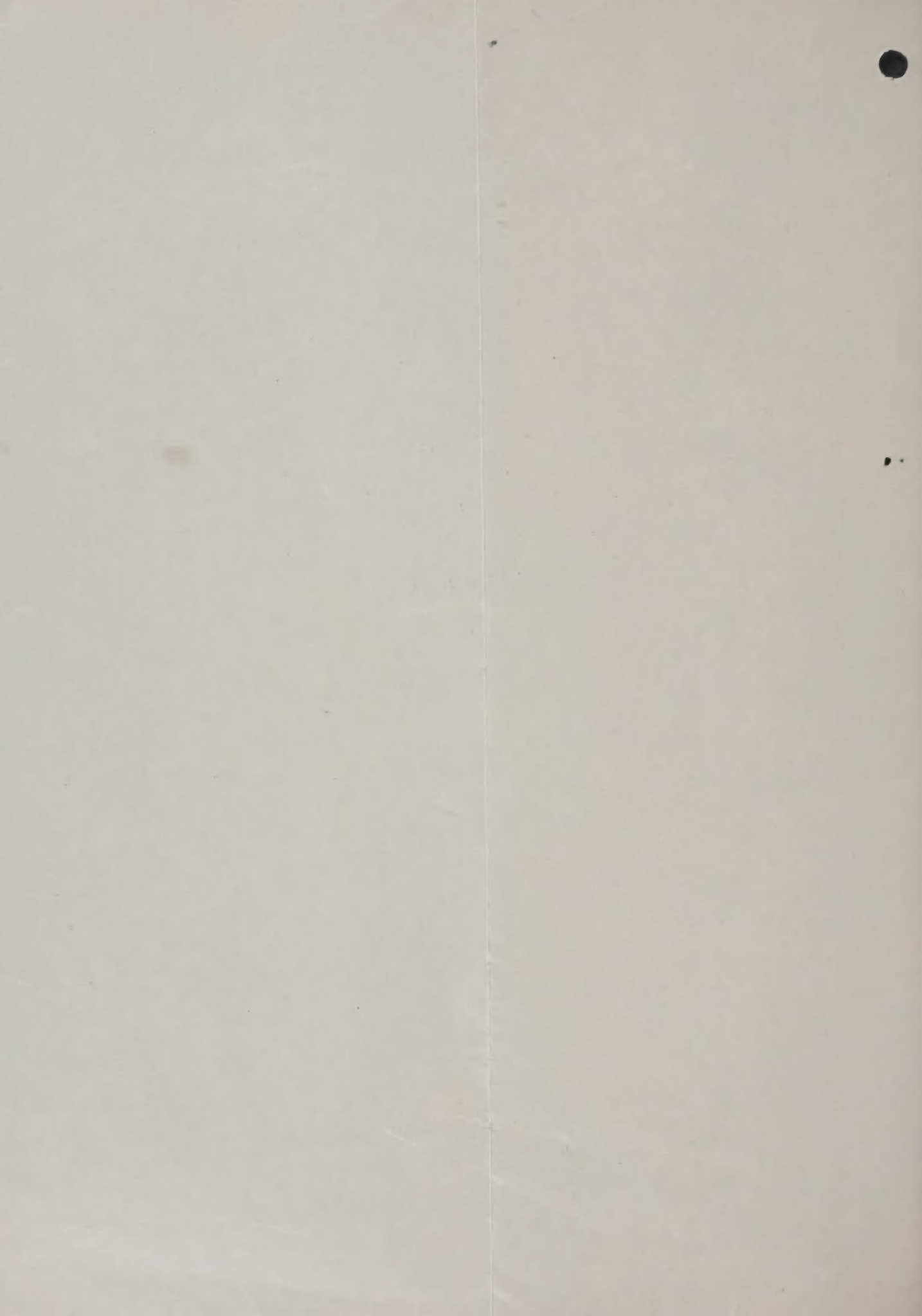
The Law of Gravitation

(with two important predictions)

BY

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The Law of Gravitation.*

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1. For a trajectory moving in a non-gravitational field

$$v^2 dt^2 = dx^2 + dy^2 + dz^2$$

or

$$\left(1 - \frac{v^2}{D^2}\right) D^2 dt^2 = D^2 dt^2 - dx^2 - dy^2 - dz^2.$$

But Einstein has assumed

$$ds^2 = \left(1 - \frac{v^2}{D^2}\right) D^2 dt^2 = \Sigma g_{\mu\nu} dx_{\mu} dx_{\nu}.$$

The postulate that this ds^2 will have the general form under gravitational influence where g 's are functions of the variable t also is fallacious. In reality, if only three variables x, y, z are transformed the g 's are independent of time. The element ds^2 is the difference of the squares of the distances travelled by the gravitational wave, and by the body unaffected by gravitation during a short interval dt . It has no other physical meaning. To keep ds real two postulates become necessary:

- (1) $ds = 0$ for light, as $v = D$,
and (2) v cannot be greater than D .

This substitution of the interval ds for the time interval dt lands us in an unreal four-dimensional world.

2. The correct method is as follows:—

For a gravitational field consisting of a spherical wave propagation from a centre

$$\frac{1}{D^2} \frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} = 0$$

$$D^2 dt^2 - dx^2 - dy^2 - dz^2 = 0,$$

where D stands for the finite velocity of gravitation.

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If by an artificial device, iDt be regarded as a variable, then both these equations become perfectly symmetrical with regard to the four variables, x, y, z, iDt . Take the change introduced by the gravitational field into the motion in the most general form

$$ds^2 = \Sigma g_{\mu\nu} dx_\mu dx_\nu,$$

where the g 's are functions of the four variables, and not their differentials. The products of the three or more differentials have been neglected as they would be too small.

Here ds is the same for a material particle and for a light corpuscle as it represents the gravitational effect of the field. It is not zero anywhere except at ∞ . It is not necessary to define ds more exactly, as it will be ultimately eliminated.

Now the equation of the path of a particle, not affected by gravitation, would be

$$v^2 dt^2 - dx^2 - dy^2 - dz^2 = 0.$$

In a short interval dt , the velocity v , unaffected by gravitation, will remain constant. Hence the most general form is

$$\Sigma a_{\mu\nu} dx_\mu dx_\nu = 0,$$

where all the a 's are constants with reference to time.

Therefore the equation of the trajectory in the gravitational field will be obtained by simply superposing the gravitational effect on this motion as

$$ds^2 = \Sigma (g_{\mu\nu} + a_{\mu\nu}) dx_\mu dx_\nu.$$

But as the effect ds^2 is perfectly symmetrical with regard to the four variables and is real, the expression on the right must be an invariant.

Now $G_{\mu\nu} = 0$ gives the g 's which determine the geometry.

If by a suitable choice of co-ordinates, the products of the differentials, except their squares, vanish in the original equation, they must also vanish in the transformed equation obtained by substitutions for only three out of the four variables. Hence the equation will retain the same form.

As the trajectory must be in a plane, and from the symmetry round the origin the coefficients should be independent of θ , we can take the equation in polar co-ordinates as

$$ds^2 = g_{11} \cdot D^2 dt^2 + g_{22} \cdot dr^2 + g_{33} \cdot r^2 d\theta^2$$

or say

$$= e^\nu \cdot D^2 dt^2 - e^\lambda \cdot dr^2 - r^2 d\theta^2,$$

where ν and λ are some functions of r only.

Following the method (purely mathematical) used by Eddington in his Relativity, § 38 (pp. 83-5) we deduce that $\lambda = -\nu$. Putting $e^\nu = \gamma$, we get

$$\gamma + \frac{d\gamma}{dr} = 1,$$

and therefore

$$\gamma = 1 - \frac{2\mu}{D^2 r},$$

where 2μ is a constant of integration.

Hence

$$ds^2 = \gamma \cdot D^2 dt^2 - \frac{1}{\gamma} dr^2 - r^2 d\theta^2.$$

Proceeding as in § 39 (pp. 85-6), we get

$$\frac{d^2\theta}{ds^2} + \frac{2}{r} \frac{dr}{ds} \frac{d\theta}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \frac{d\nu}{dr} \frac{dr}{ds} \frac{dt}{ds} = 0.$$

These two on integration give

$$r^2 \frac{d\theta}{ds} = \frac{h}{D} \quad \text{and} \quad \frac{dt}{ds} = A e^{-\nu} = \frac{A}{\gamma},$$

where h and A are constants.

Eliminating ds and dt , we get

$$\frac{1}{\gamma} \left(\frac{h}{Dr^2} \frac{dr}{d\theta} \right)^2 + \frac{h^2}{D^2 r^2} - \frac{A^2}{\gamma} = -1.$$

Substituting the value of γ we get

$$\left(\frac{1}{r^2} \frac{dr}{d\theta} \right)^2 + \frac{1}{r^2} = \frac{(A^2 - 1) D^2}{h^2} + \frac{2\mu}{h^2 r} + \frac{2\mu}{D^2 r^3}$$

or

$$\left(\frac{du}{d\theta} \right)^2 + u^2 = \frac{(A^2 - 1) D^2}{h^2} + \frac{2\mu}{h^2} u + \frac{2\mu}{D^2} u^3$$

$\therefore$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{3\mu}{D^2} u^2.$$

This is the equation of motion both for light and matter.

3. A simpler though less rigorous method is as follows :—

The equation for motion of the particle unaffected by gravitation in a plane is

$$dx^2 + dy^2 - v^2 dt^2 = 0,$$

where v is the initial velocity at the point (x, y) , unaffected by gravitation. Suppose that as a result of the gravitational influence during the short interval dt , in which v is constant, the equation becomes

$$(1+P)dx^2 + (1+Q)dy^2 + \left(R - \frac{v^2}{D^2} \right) D^2 dt^2 = 0,$$

where P , Q and R may be some functions of x , y and t .

Let us transform only two of these three variables by putting

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

Then the equation becomes

$$(1 + P \cos^2 \theta + Q \sin^2 \theta) dr^2 + (1 + P \sin^2 \theta + Q \cos^2 \theta) r^2 d\theta^2 + 2(Q - P) r \sin \theta \cos \theta dr d\theta + \left(R - \frac{v^2}{D^2} \right) D^2 dt^2 = 0,$$

where P , Q and R are now symmetrical functions of $r \cos \theta$, $r \sin \theta$ and iDt . For this to be symmetrical with respect to the three variables the coefficient of $dr \cdot d\theta$ must be zero, as other products do not occur. Hence $Q = P$. Therefore the equation takes the form

$$(1 + P) dr^2 + (1 + P) r^2 d\theta^2 + \left(R - \frac{v^2}{D^2} \right) D^2 dt^2 = 0.$$

Here on account of symmetry and as a particular case R may be equal to P , but is not equal to $-P$, as that would make the gravitational influence vanish, which it does not except at ∞ .

Now (1) from the symmetry round the origin, these coefficients must be independent of θ .

(2) As the source is not varying with time, and in the wave expansion t depends on r , these must be functions of r only.

(3) Also as the effect must be zero at ∞ , the coefficients when expanded give

$$P = \sum_1^{\infty} \frac{A_n}{r^n} \quad \text{and} \quad R = \sum_1^{\infty} \frac{B_n}{r^n},$$

where the A 's and B 's are constants.

Therefore

$$\left(1 + \sum_1^{\infty} \frac{A_n}{r^n} \right) dr^2 + \left(1 + \sum_1^{\infty} \frac{A_n}{r^n} \right) r^2 d\theta^2 + \left(\sum_1^{\infty} \frac{B_n}{r^n} - \frac{v^2}{D^2} \right) D^2 dt^2 = 0.$$

Substituting $r^2 \frac{d\theta}{dt} = h$ nearly, we get

$$\left(1 + \frac{A_1}{r} + \frac{A_2}{r^2} + \dots \right) dr^2 + \left(1 + \frac{A_1}{r} + \frac{A_2}{r^2} + \dots \right) r^2 d\theta^2 + \left(\frac{B_1}{r} + \frac{B_2}{r^2} + \dots - \frac{v^2}{D^2} \right) \frac{D^2}{h^2} \cdot r^4 d\theta^2 = 0.$$

But for the orbits of the planets and comets, and of light round the Sun, both r and h are very large and h is comparable to $r \cdot D$. And so as an approximation terms up to $\frac{1}{r^3}$ and $\frac{D^2}{h^2} \frac{1}{r}$ only need be retained.

Hence
$$\left(\frac{1}{r^2} \frac{dr}{d\theta}\right)^2 + \left(1 + \frac{A_1}{r}\right) \frac{1}{r^2} = \frac{v^2}{h^2} - \frac{D^2 B_1}{h^2} \cdot \frac{1}{r}$$

or
$$\left(\frac{du}{d\theta}\right)^2 + (1 + A_1 u) u^2 = \frac{v^2}{h^2} - \frac{D^2 B_1}{h^2} \cdot u.$$

Therefore
$$\begin{aligned} \frac{d^2 u}{d\theta^2} + u &= \frac{-D^2 B_1}{2h^2} - \frac{3A_1}{2} u^2 \\ &= \frac{GM}{h^2} + k \cdot \frac{3GM}{D^2} u^2, \end{aligned}$$

where M is the mass of the gravitating body, and $G = -\frac{D^2 B_1}{2M}$, a gravitational constant, and k is some other constant.

From the observation of the advance of the perihelion of Mercury (and without assuming the particular case that $R = P$ which it really must be on account of symmetry) it is known that k is nearly unity.

Hence
$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2} + \frac{3\mu}{D^2} u^2.$$

This can be transformed into

$$\frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r^2} - \frac{3\mu h^2}{D^2} \cdot \frac{1}{r^4}.$$

In this law of gravitational attraction the inverse cube of the distance does not occur.

4. Even in Einstein's Relativity, as ds and dt are ultimately eliminated, the equation of the orbit should be the same for a material particle as well as a light corpuscle, and the constant of integration h should remain finite for the velocity of light also.

But Einstein's postulate has compelled him to take $ds = 0$ and therefore $h = \infty$ for light, with the result that for light the equation becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{3\mu}{D^2} u^2.$$

The New Theory gives the same equation both for matter and light and h remains finite. This difference furnishes the most crucial and decisive tests for the two rival theories to be verified by observation.

5. The ratio of the wave-length of light from the edge of the Solar disc to that on the earth has been shown by the author to be

$$= 1 + \frac{\mu}{D^2} \int_{\infty}^a \left(-\frac{1}{r^2} - \frac{3h^2}{D^2} \cdot \frac{1}{r^4} \right) dr, \text{ nearly,}$$

where a is the radius of the Sun.

$$= 1 + \frac{2\mu}{D^2 a}, \text{ as } h = Da, \text{ nearly.}$$

Hence the spectral shift $\frac{2\mu}{D^2a}$ is double of Einstein's value.

6. (1) The *minimum* deflection of light from Stars past the Sun has been obtained by putting $h = rD$, as h cannot be greater than $r \cdot D$.

This yields a deflection $4/3$ times Einstein's value, i.e.

$2''.32$ instead of $1''.745$.

(2) The *maximum* deflection of light has been shown to be

$$= -\frac{1}{D^2R} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ -\frac{\mu}{r^2} - \frac{3\mu h^2}{D^2} \cdot \frac{1}{r^4} \right\} \cos \psi \, d\psi,$$

where $r \cos \psi = R$, the shortest distance from the Sun, and $h = DR$ nearly.

$$= -\frac{6\mu}{D^2R}, \text{ which is } 3/2 \text{ times Einstein's value, i.e. } 2''.61.$$

As the deviation from the straight path is small, the real value of the deflection is nearer the maximum than the minimum.

TWO PREDICTIONS.

(1) The spectral shift of light from the edge of the Solar disc is 100% more than Einstein's value.

(2) The deflection of light from Stars past the Sun is between 42 to 50% more than Einstein's value.

